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# Finite polynomial cohomology for general varieties

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*In honour of Glenn Stevens' 60th birthday*

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**Abstract** Nekovář and Nizioł (Syntomic cohomology and  $p$ -adic regulators for varieties over  $p$ -adic fields, 2013) have introduced in a version of syntomic cohomology valid for arbitrary varieties over  $p$ -adic fields. This uses a mapping cone construction similar to the rigid syntomic cohomology of (Besser, Israel J Math 120(1):291–334, 2000) in the good-reduction case, but with Hyodo–Kato (log-crystalline) cohomology in place of rigid cohomology. In this short note, we describe a cohomology theory which is a modification of the theory of Nekovář–Nizioł, modified by replacing  $1 - \varphi$  with other polynomials in  $\varphi$ . This is the analogue for bad-reduction varieties of the finite-polynomial cohomology of (Besser, Invent Math 142(2):397–434, 2000); and we use this cohomology theory to give formulae for  $p$ -adic regulator maps, extending the results of (Besser, Invent Math 142(2):397–434, 2000; Besser, Israel J Math 120(1):335–360, 2000; Besser, Israel J Math 190(1):29–66, 2012) to varieties over  $p$ -adic fields, without assuming any good reduction hypotheses.

**Keywords** Finite-polynomial cohomology · Syntomic cohomology · Regulators

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**Résumé** Nekovar et Niziol (Syntomic cohomology and  $p$ -adic regulators for varieties over  $p$ -adic fields, 2013) ont introduit une version de la cohomologie syntomique pour les variétés définies sur un corps  $p$ -adique. Leur construction généralise la cohomologie rigide syntomique de (Besser, Israel J Math 120(1):291–334, 2000) dans le cas de bonne réduction, remplaçant la cohomologie rigide par la cohomologie de Hyodo-Kato. Nous décrivons une modification de la théorie de Nekovar-Niziol, remplaçant l’opérateur  $1-\Phi$  par d’autres polynômes en  $\Phi$ . Ceci est l’analogue de la “finite-polynomial cohomology” de (Besser, Invent. Math 142(2):397–434, 2000) pour les variétés de mauvaise réduction. Nous utilisons cette cohomologie pour obtenir des formules pour les régulateurs  $p$ -adiques, généralisant les résultats de (Besser, Invent. Math 142(2):397–434, 2000; Besser, 11 Israel J Math 120(1):335–360, 2000; Besser, Israel J Math 190(1):29–66, 2012) dans le cas de bonne réduction.

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## 1 Preliminaries from $p$ -adic Hodge theory

### 1.1 Filtered $(\varphi, N)$ -modules and their cohomology

We recall some standard constructions for  $(\varphi, N)$ -modules, following §2.4 of [10]. Let  $K$  be a  $p$ -adic field (i.e. the field of fractions of a complete DVR  $V$  of mixed characteristic  $(0, p)$ , with perfect residue field  $k$ ).

**Definition 1.1.1** We define a filtered  $(\varphi, N, G_K)$ -module over  $K$  to be the data of a finite-dimensional  $K_0^{\text{nr}}$ -vector space  $D$ , where  $K_0$  is the maximal unramified subfield of  $K$  and  $K_0^{\text{nr}}$  its maximal unramified extension, equipped with the following structures:

- a  $K_0^{\text{nr}}$ -semilinear, bijective Frobenius  $\varphi$ ;
- a  $K_0^{\text{nr}}$ -linear monodromy operator  $N$  satisfying  $N\varphi = p\varphi N$ ;
- a  $K_0^{\text{nr}}$ -semilinear action of  $G_K$ ;
- and a decreasing, separated, exhaustive filtration of the  $K$ -vector space  $D_K = (D \otimes_{K_0^{\text{nr}}} \overline{K})^{G_K}$  by  $K$ -vector subspaces  $\text{Fil}^i D_K$ .

These are the objects of a pre-abelian category  $\text{MF}_{(\varphi, N, G_K)}$ . (It is not an abelian category, since morphisms may not be strictly compatible with the filtration.)

**Theorem 1.1.2** (Colmez–Fontaine) *If  $K$  is a finite extension of  $\mathbf{Q}_p$ , the subcategory of “weakly admissible” filtered  $(\varphi, N, G_K)$ -modules is equivalent to the category of potentially semistable  $\mathbf{Q}_p$ -linear representations of  $G_K$ , via the functor  $V \mapsto \mathbf{D}_{\text{pst}}(V) = \bigcup_M (V \otimes \mathbf{B}_{\text{st}})^{G_M}$ , where the direct limit is taken over finite extensions  $M/K$ .*

We write  $D_{\text{st}} = D^{G_K}$ , which is a  $K_0$ -vector space. We say that  $D$  is *semistable* if the  $G_K$ -action on  $D$  is unramified, in which case we have

$$D = D_{\text{st}} \otimes_{K_0} K_0^{\text{nr}} \quad \text{and} \quad D_K = D_{\text{st}} \otimes_{K_0} K;$$

and we say  $D$  is *crystalline* if it is semistable with  $N = 0$ . These notions are compatible with the usual ones for weakly admissible  $D$  via the Colmez–Fontaine theorem.

**Definition 1.1.3** Let  $D$  be a filtered  $(\varphi, N, G_K)$ -module and define a complex  $C_{\text{st}}^\bullet(D)$  by

$$D_{\text{st}} \oplus \text{Fil}^0 D_K \longrightarrow D_{\text{st}} \oplus D_{\text{st}} \oplus D_K \longrightarrow D_{\text{st}},$$

with morphisms given by  $(u, v) \mapsto ((1 - \varphi)u, Nu, u - v)$  and  $(w, x, y) \mapsto Nw - (1 - p\varphi)x$ .

**Theorem 1.1.4** (Bloch–Kato, Nekovář) *The complex  $C_{\text{st}}^\bullet$  computes Ext groups in the category of filtered  $(\varphi, N, G_K)$ -modules; the subcategory of weakly admissible objects is closed under extensions, and the functor  $\mathbf{D}_{\text{pst}}$  induces functorial maps*

$$H_{\text{st}}^i(D) \rightarrow H^i(K, V).$$

These maps are isomorphisms for  $i = 0$  and injective for  $i = 1$ .

## 1.2 Variants

We now construct a variant of the complex  $C_{\text{st}}(D)$  in which the semilinear Frobenius is replaced by a “partially linearized” one, and  $1 - \varphi$  by a more general polynomial.

We choose a finite extension  $L/\mathbf{Q}_p$  contained in  $K$ , and we write  $f = [L_0 : \mathbf{Q}_p]$  and  $q = p^f$ . We can then define  $D_{\text{st},L} = D_{\text{st}} \otimes_{L_0} L$ , which we equip with an  $L$ -linear operator  $\Phi$  given by extending scalars from the  $L_0$ -linear operator  $\varphi^f$  on  $D_{\text{st}}$ .

**Definition 1.2.1** Let  $P \in 1 + TL[T]$  be a polynomial. We write  $C_{\text{st},L,P}^\bullet(D)$  for the complex of  $L$ -vector spaces

$$D_{\text{st},L} \oplus \text{Fil}^0 D_K \longrightarrow D_{\text{st},L} \oplus D_{\text{st},L} \oplus D_K \longrightarrow D_{\text{st},L},$$

with the maps given by  $u, v \mapsto (P(\Phi)u, Nu, u - v)$  and  $(w, x, y) \mapsto Nw - P(q\Phi)x$ .

Thus  $C_{\text{st}}^\bullet(D) = C_{\text{st},\mathbf{Q}_p,P_0}^\bullet(D)$  where  $P_0(T) = 1 - T$ . We write  $H_{\text{st},L,P}^\bullet(D)$  for the cohomology of the complex  $C_{\text{st},L,P}^\bullet(D)$ .

**Remark 1.2.2** In applications of the theory, we are almost always interested in the case when  $K$  is finite over  $\mathbf{Q}_p$  and  $L = K$ ; but in order to set up the theory we need a base-change compatibility which seems to be easier to prove for varying  $K$  but fixed  $L$ , which is why we have set up the theory for  $L \neq K$ .

We have natural compatibilities when we change the polynomial  $P$  or the field  $L$ :

**Definition 1.2.3** If  $P, Q \in 1 + TL[T]$ , we define natural transformations  $C_{\text{st},L,P}^\bullet(-) \rightarrow C_{\text{st},L,PQ}^\bullet(-)$  via

$$\begin{array}{ccccc} D_{\text{st},L} \oplus \text{Fil}^0 D_K & \rightarrow & D_{\text{st},L} \oplus D_{\text{st},L} \oplus D_K & \longrightarrow & D_{\text{st},L} \\ \text{(id, id)} \downarrow & & (Q(\Phi), \text{id, id}) \downarrow & & Q(q\Phi) \downarrow \\ D_{\text{st},L} \oplus \text{Fil}^0 D_K & \rightarrow & D_{\text{st},L} \oplus D_{\text{st},L} \oplus D_K & \longrightarrow & D_{\text{st},L} \end{array}$$

**Definition 1.2.4** Suppose  $L \subset L'$  are two subfields of  $K$  finite over  $\mathbf{Q}_p$ , let  $P' \in 1 + TL[T]$ , and define  $P(T) = P'(T^d)$  where  $d = [L'_0 : L_0] = f'/f$ . Then we have natural transformations  $C_{\text{st},L,P}^\bullet(-) \rightarrow C_{\text{st},L',P'}^\bullet(-)$  arising from the inclusions

$$D_{\text{st}} \otimes_{L_0} L \rightarrow D_{\text{st}} \otimes_{L'_0} L'.$$

Note that the map of Definition 1.2.4 is an isomorphism if  $L'/L$  is an unramified extension.

**Remark 1.2.5** One can define, in the obvious fashion, a pre-abelian category  $\mathbf{MF}_{(\Phi, N, G_K, L)}$  of “filtered  $(\Phi, N, G_K)$ -modules with coefficients in  $L$ ”, whose objects are finite-dimensional  $LK_0^{\text{nr}}$ -vector spaces  $D_L$  with an  $L$ -linear  $q$ -th power Frobenius  $\Phi$ , a monodromy operator  $N$ , a  $G_K$ -action, and a filtration on  $D_K = (D_L \otimes_{LK_0^{\text{nr}}} \bar{K})^{G_K}$ . There is a natural extension-of-coefficients functor

$$\mathbf{MF}_{(\varphi, N, G_K)} \rightarrow \mathbf{MF}_{(\Phi, N, G_K, L)},$$

and the functors  $C_{\text{st}, L, P}^\bullet(-)$  naturally factor through this.

For each polynomial  $P \in 1 + TL[T]$ , one can define an object  $\mathbf{1}_P$  of  $\mathbf{MF}_{(\Phi, N, G_K, L)}$  by taking  $\mathbf{1}_P = L[X]/P(X)$ , with  $\Phi$  acting as multiplication by  $X$  (and  $N$  and  $G_K$  acting trivially, and filtration concentrated in degree 0). Then the complex  $C_{\text{st}, L, P}^\bullet(D)$  computes the Ext groups  $\text{Ext}^i(\mathbf{1}_P, D)$  in this category, and the change-of- $P$  map of Definition 1.2.3 is then obtained by contravariant functoriality from the obvious map  $\mathbf{1}_{PQ} \rightarrow \mathbf{1}_P$ . Note, however, that  $\mathbf{1}_P$  is not weakly admissible in general.

### 1.3 Cup products

**Definition 1.3.1** (cf. [4, Definition 4.1]) If  $P, Q \in 1 + TL[T]$ , then we define  $P \star Q \in 1 + TL[T]$  as the polynomial with roots  $\{\alpha_i \beta_j\}$ , where  $\{\alpha_i\}$  and  $\{\beta_j\}$  are the roots of  $P$  and  $Q$  respectively.

**Proposition 1.3.2** *There are cup-products  $C_{\text{st}, L, P_1}^\bullet(D_1) \times C_{\text{st}, L, P_2}^\bullet(D_2) \rightarrow C_{\text{st}, L, P_1 \star P_2}^\bullet(D_1 \otimes D_2)$ , associative and graded-commutative up to homotopy, compatible with the change-of- $P$  maps of Definition 1.2.3 and the change-of- $L$  maps of Definition 1.2.4.*

*Proof* This is a straightforward exercise in homological algebra. Let  $\lambda \in K$  and choose polynomials  $a(T_1, T_2)$  and  $b(T_1, T_2)$  such that  $a(T_1, T_2)P_1(T_1) + b(T_1, T_2)P_2(T_2) = (P_1 \star P_2)(T_1 T_2)$ . Then the cup-products in the various degrees are given in Table 1. One verifies easily that changing the value of  $\lambda$ , or the polynomials  $a$  and  $b$ , changes the product by a chain homotopy.  $\square$

### 1.4 Convenient modules

**Definition 1.4.1** Let  $D$  be a filtered  $(\varphi, N, G_K)$ -module. We say  $D$  is *convenient* (for some choice of  $L$  and  $P$ ) if  $D$  is crystalline and  $P(\Phi)$  and  $P(q\Phi)$  are bijective as endomorphisms of  $D_{\text{st}, L}$ .

**Table 1** Cup-products on the complexes  $C_{\text{st}, L, P}^\bullet$

	$(u', v')$	$(w', x', y')$	$z'$
$(u, v)$	$(u \otimes u', v \otimes v')$	$\begin{pmatrix} b(\Phi_1, \Phi_2)(u \otimes w'), \\ u \otimes x', \\ (\lambda u + (1 - \lambda)v) \otimes y' \end{pmatrix}$	$b(\Phi_1, q\Phi_2)(u \otimes z')$
$(w, x, y)$	$\begin{pmatrix} a(\Phi_1, \Phi_2)(w \otimes u'), \\ x \otimes u', \\ y \otimes ((1 - \lambda)u' + \lambda v') \end{pmatrix}$	$-a(\Phi_1, q\Phi_2)(w \otimes x') + b(q\Phi_1, \Phi_2)(x \otimes w')$	0
$z$	$a(q\Phi_1, \Phi_2)(z \otimes u')$	0	0

Note that  $D$  is convenient if and only if  $D^*(1)$  is convenient. If  $D$  is convenient, then the inclusion  $D_K \hookrightarrow C_{\text{st},L,P}^1(D)$  induces an isomorphism

$$\frac{D_K}{\text{Fil}^0 D_K} \cong H_{\text{st},L,P}^1(D). \quad (1)$$

The inverse of this isomorphism is given by

$$(w, x, y) \mapsto y - \iota(P(\Phi)^{-1}w) \pmod{\text{Fil}^0}, \quad (2)$$

where  $\iota$  is the natural inclusion  $D_{\text{st},L} \hookrightarrow D_K$ ; note that we must have  $Nw = P(q\Phi)x$  since  $(w, x, y)$  is a cocycle, but by assumption  $N = 0$  and  $P(q\Phi)$  is bijective, so in fact we have  $x = 0$ . Note that this map commutes with the change-of- $P$  and change-of- $L$  maps (where defined).

The following special case will be of importance below:

**Proposition 1.4.2** *If  $P(1) \neq 0$  and  $P(q^{-1}) \neq 0$ , then there is an isomorphism*

$$H_{\text{st},L,P}^1(\mathbf{Q}_p(1)) \cong K.$$

The key to our description of syntomic regulators is the following. We now take  $L = K$ .

**Proposition 1.4.3** *Suppose that  $D$  is convenient (for  $L = K$  and some polynomial  $P \in 1 + TK[T]$ ). Then for any  $\lambda \in \text{Fil}^0 D^*(1)_K = (D_K / \text{Fil}^0)^*$ , there is a polynomial  $Q \in 1 + XK[X]$  such that  $\lambda \in H_{\text{st},K,Q}^0(D^*(1))$  and  $(P \star Q)(1) \neq 0$ ,  $(P \star Q)(q^{-1}) \neq 0$ ; and if  $P$  is such a polynomial, then we have a commutative diagram*

$$\begin{array}{ccc} D_K / \text{Fil}^0 & \xrightarrow[\cong]{} & H_{\text{st},K,P}^1(D) \\ \lambda \downarrow & & \downarrow \cup \lambda \\ K & \xlongequal{\quad} & H_{\text{st},K,P \star Q}^1(\mathbf{Q}_p(1)) \end{array}$$

where the right-hand vertical map is the cup-product of the previous section.

**Remark 1.4.4** The utility of this proposition is that if  $D$  is convenient, it allows us to completely describe an element of  $H_{\text{st},L,P}^1(D)$  via its cup-products with elements of  $H_{\text{st},L,Q}^0(D^*(1))$  for suitable polynomials  $Q$ . This is the key to our computations of regulator maps below. Note that even if  $P$  is some very simple polynomial (such as  $1 - T$ ), we still need to be able to choose  $Q$  freely for this to work.

## 2 Finite polynomial cohomology for general varieties

### 2.1 Summary of the theory of Nekovář–Nizioł

We briefly summarize some of the main results of the paper [10]. Let  $\text{Var}(K)$  be the category of varieties over  $K$  (i.e. reduced separated  $K$ -schemes of finite type).

**Theorem 2.1.1** *There exists a functor  $X \mapsto \mathbb{R}\Gamma_{\text{syn}}(X_h, *)$  from  $\text{Var}(K)$  to the category of graded-commutative differential graded  $E_\infty$ -algebras over  $\mathbf{Q}_p$ , equipped with period morphisms*

$$\rho_{\text{syn}} : \mathbb{R}\Gamma_{\text{syn}}(X_h, r) \rightarrow \mathbb{R}\Gamma_{\text{ét}}(X_{K,\text{ét}}, \mathbf{Q}_p(r)).$$

The cohomology theory  $X \mapsto \mathbb{R}\Gamma_{\text{syn}}(X_h, r)$  has pushforward maps for projective morphisms, and has a functorial map from Voevodsky's motivic cohomology, compatible with the étale realization map via  $\rho_{\text{syn}}$ .

For  $X$  a variety over  $K$ , let  $H_{\text{HK}}^j(X_h)$  and  $H_{\text{dR}}^j(X_h)$  be the extensions of Hyodo–Kato and de Rham cohomologies defined by Beilinson [1], and  $\iota_{\text{dR}}^B : H_{\text{HK}}^j(X_h) \otimes_{K_0} K \rightarrow H_{\text{dR}}^j(X_h)$  the comparison morphism relating them (which is an isomorphism if  $X$  has a semistable model over  $O_K$ ).

The main result of [1] shows that we have

$$H_{\text{HK}}^j(X_h) = D^j(X_h)_{\text{st}}, \quad H_{\text{dR}}^j(X_h) = D^j(X_h)_K,$$

where  $D^j(X_h)$  is the filtered  $(\varphi, N, G_K)$ -module given by

$$D^j(X_h) = \varinjlim_{K'} H_{\text{HK}}^j(X_{K',h}),$$

where  $K'$  varies over finite extensions of  $K$ , and that there is a canonical isomorphism of  $(\varphi, N, G_K)$ -modules

$$D^j(X_h) \cong \mathbf{D}_{\text{pst}} \left( H^j(X_{\overline{K}, \text{ét}}, \mathbf{Q}_p) \right).$$

**Theorem 2.1.2** *There is a “syntomic descent spectral sequence”*

$$H_{\text{st}}^i(D^j(X_h)(r)) \Rightarrow H_{\text{syn}}^{i+j}(X_h, r),$$

and the morphisms

$$H_{\text{st}}^i(D^j(X_h)(r)) \rightarrow H^i(K, H_{\text{ét}}^j(X_{\overline{K}, \text{ét}}, \mathbf{Q}_p(r)))$$

from Theorem 1.1.4 assemble into a morphism from the syntomic descent spectral sequence to the Hochschild–Serre spectral sequence  $H^i(K, H_{\text{ét}}^j(X_{\overline{K}, \text{ét}}, \mathbf{Q}_p(r))) \Rightarrow H_{\text{ét}}^{i+j}(X_{K, \text{ét}}, \mathbf{Q}_p(r))$ , compatible with the period morphism  $\rho_{\text{syn}}$  on the abutment.

## 2.2 Definition of $P$ -syntomic cohomology

We now develop a theory which very closely imitates that of [10], but modified to use general polynomials in Frobenius in the place of  $1 - \varphi$ , and replacing the  $p$ -power Frobenius  $\varphi$  with a “partially linearized” Frobenius. We choose a finite extension  $L/\mathbf{Q}_p$  is a finite extension contained in  $K$ , with residue class degree  $f = [L_0 : \mathbf{Q}_p]$ , and  $P \in 1 + TL[T]$ .

Let  $(U, \overline{U})$  be an “arithmetic pair” (in the sense of *op.cit.*), log-smooth over  $V^\times$  (i.e.  $\text{Spec}(V)$  with the log structure associated to the closed point).

Let  $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$  be the rational crystalline cohomology (defined as

$$\mathbf{Q}_p \otimes_{\mathbf{Z}_p} \text{holim}_n \mathbb{R}\Gamma \left( \overline{U}_{\text{ét}}, Ru_{U_n^\times/W_n(k)*} \mathcal{O}_{U_n^\times/W_n(k)} \right),$$

cf. Sect. 3.1 of *op.cit.*). This is a complex of  $K_0$ -vector spaces, hence of  $L_0$ -vector spaces, and we write  $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L = L \otimes_{L_0} \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$ . There is a  $K_0$ -semilinear Frobenius  $\varphi$  on  $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$ , and we let  $\Phi = \varphi^f$ , which is  $L_0$ -linear and thus extends to an  $L$ -linear operator on  $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L$ . For  $r \geq 0$  we write  $\Phi_r = q^{-r} \Phi = (p^{-r} \varphi)^f$ .

**Remark 2.2.1** Curiously, the complex  $\mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}}$  is already a complex of  $K$ -vector spaces, but this  $K$ -linear structure interacts very badly with the Frobenius, and passing to Frobenius eigenspaces “kills off” all contributions from  $K \setminus K_0$ . So we must “add the  $K$ -linear structure a second time” in order to obtain a  $K$ -linearized theory of syntomic cohomology.

For any  $r \geq 0$ , the quasi-isomorphism  $\gamma_r^{-1} : \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U}, \mathcal{O}/\mathcal{I}^{[r]})_{\mathbf{Q}} \xrightarrow{\sim} \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r$  gives rise to a morphism (not, of course, a quasi-isomorphism)

$$\gamma_{r,L}^{-1} : \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L \rightarrow \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r.$$

**Definition 2.2.2** We define

$$\mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r) = \left[ \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L \xrightarrow{P(\Phi_r), \gamma_{r,L}^{-1}} \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r \right]$$

*Remark 2.2.3* Note that the use  $P(\Phi_r)$ , rather than  $P(\Phi)$ ; this choice gives somewhat cleaner formulations of some results (e.g. the pushforward maps and the syntomic descent spectral sequence), but has the disadvantage of introducing a notational discrepancy between  $\mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r)$  and the finite-polynomial cohomology of [4].

The complexes  $\mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r)$  are complexes of  $L$ -vector spaces, functorial in pairs  $(U, \overline{U})$ , equipped with  $L$ -linear cup-products

$$\mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r) \times \mathbb{R}\Gamma_{\text{syn},L,Q}(U, \overline{U}, s) \longrightarrow \mathbb{R}\Gamma_{\text{syn},L,P \star Q}(U, \overline{U}, r+s)$$

which are associative and graded-commutative, up to coherent homotopy (i.e. the direct sum

$$\bigoplus_{r,P} \mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r)$$

is an  $E_{\infty}$ -algebra over  $L$ ). These cup-products are given explicitly by the same recipe as in 1.3.2 above.

The following properties are immediate:

**Proposition 2.2.4** (1) (*Compatibility with syntomic cohomology*) If  $L = \mathbf{Q}_p$  and  $P(T)$  is the polynomial  $1 - T$ , then  $\mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r)$  coincides with  $\mathbb{R}\Gamma_{\text{syn}}(U, \overline{U}, r)_{\mathbf{Q}}$  as defined in [10].

(2) (*Change of  $P$* ) If  $P, Q$  are two polynomials, there is a map

$$\mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r)_{\mathbf{Q}} \rightarrow \mathbb{R}\Gamma_{\text{syn},L,PQ}(U, \overline{U}, r)_{\mathbf{Q}}$$

functorial in  $(U, \overline{U})$  and compatible with cup-products, which is given by the diagram

$$\begin{array}{ccc} \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L & \xrightarrow{P(\Phi_r), \gamma_{r,L}^{-1}} & \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r \\ \text{id} \downarrow & & \downarrow (Q(\Phi_r), \text{id}) \\ \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L & \xrightarrow{PQ(\Phi_r), \gamma_{r,L}^{-1}} & \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r \end{array}$$

(3) (*Change of  $L$* ) Let  $L'/L$  be a finite extension, and suppose that we have  $P(T) = P'(T^d)$  for some polynomial  $P'$ , where  $d = [L'_0 : L_0] = f'/f$ . Then there is a natural map

$$\mathbb{R}\Gamma_{\text{syn},L,P}(U, \overline{U}, r) \longrightarrow \mathbb{R}\Gamma_{\text{syn},L',P'}(U, \overline{U}, r),$$

functorial in  $(U, \overline{U})$  and compatible with cup-products. If  $L'/L$  is an unramified extension, this is an isomorphism.



*Proof* (1) and (2) are obvious by construction. For (3), note that we have  $\Phi'_r = (p^{-r}\varphi)^{f'}$  and  $\Phi_r = (p^{-r}\varphi)^f$ , so  $P(\Phi_r) = P'(\Phi'_r)$ . Hence we have a commutative diagram

$$\begin{array}{ccc} \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L & \xrightarrow{P(\Phi_r), \gamma_{r,L}^{-1}} & \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{L'} & \xrightarrow{P'(\Phi'_r), \gamma_{r,L'}^{-1}} & \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{L'} \oplus \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r \end{array}$$

where the vertical maps are the natural maps induced from the inclusion  $L \subset L'$ .

If  $L'/L$  is unramified, so  $L' = LL'_0$ , then we have a canonical isomorphism of complexes of  $L'_0$ -vector spaces

$$L \otimes_{L_0} \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}} \cong L' \otimes_{L'_0} \mathbb{R}\Gamma_{\text{cr}}(U, \overline{U})_{\mathbf{Q}},$$

so the vertical maps are isomorphisms.

We will be particularly interested in a special case of this:

**Proposition 2.2.5** *If  $P(1) = 0$  then there is a morphism of complexes*

$$\mathbb{R}\Gamma_{\text{syn}}(U, \overline{U}, r)_{\mathbf{Q}} \rightarrow \mathbb{R}\Gamma_{\text{syn}, L, P}(U, \overline{U}, r),$$

*functorial in  $(U, \overline{U})$  and compatible with cup-products.*

*Proof* This is built up by combining all three parts of the above proposition. Firstly, (1) identifies Nekovář–Nizioł’s cohomology  $\mathbb{R}\Gamma_{\text{syn}}(U, \overline{U}, r)_{\mathbf{Q}}$  with the special case  $P(T) = 1 - T$ ,  $L = \mathbf{Q}_p$  of our construction. If  $P$  is any polynomial with  $P(1) = 0$ , then  $P(T^f)$  is divisible by  $1 - T$ , so (2) gives a morphism

$$\mathbb{R}\Gamma_{\text{syn}, \mathbf{Q}_p, 1-T}(U, \overline{U}, r) \rightarrow \mathbb{R}\Gamma_{\text{syn}, \mathbf{Q}_p, P(T^f)}(U, \overline{U}, r)_{\mathbf{Q}_p}.$$

Finally (3) gives a morphism

$$\mathbb{R}\Gamma_{\text{syn}, \mathbf{Q}_p, P(T^f)}(U, \overline{U}, r) \rightarrow \mathbb{R}\Gamma_{\text{syn}, L, P}(U, \overline{U}, r).$$

All of these are visibly functorial in pairs  $(U, \overline{U})$ .

We now show a relation analogous to Proposition 3.7 of *op.cit.*. Suppose that  $(U, \overline{U})$  is of Cartier type, so Hyodo–Kato cohomology is defined. We use Beilinson’s variant of Hyodo–Kato cohomology, which has the advantage of having a comparison map  $\iota_{\text{dR}}^B : \mathbb{R}\Gamma_{\text{HK}}^B(U, \overline{U})_K \xrightarrow{\sim} \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)$  which is defined at the level of complexes and does not depend on making a choice of uniformizer of  $K$ .

**Proposition 2.2.6** *For each uniformizer  $\pi$  of  $V$ , there is a quasi-isomorphism*

$$\mathbb{R}\Gamma_{\text{syn}, L, P}(U, \overline{U}, r) \xrightarrow[\sim]{\alpha'_{\pi, P, L}} \left[ \begin{array}{ccc} \mathbb{R}\Gamma_{\text{HK}}^B(U, \overline{U})_L & \xrightarrow{(P(\Phi_r), \iota_{\text{dR}}^B)} & \mathbb{R}\Gamma_{\text{HK}}^B(U, \overline{U})_L \oplus \mathbb{R}\Gamma_{\text{dR}}(U, \overline{U}_K)/\text{Fil}^r \\ N \downarrow & & (N, 0) \downarrow \\ \mathbb{R}\Gamma_{\text{HK}}^B(U, \overline{U})_L & \xrightarrow{P(\Phi_{r-1})} & \mathbb{R}\Gamma_{\text{HK}}^B(U, \overline{U})_L \end{array} \right]$$

where  $q = p^f$ .

**Remark 2.2.7** This map does actually depend on the choice of a uniformizer  $\pi$ , although its source and target are independent of any such choice.

*Proof* We follow exactly the same argument as the proof given in *op.cit.*. It suffices to check that  $P(\Phi)$  is invertible on  $I \otimes_{W(k)} H_{\text{HK}}^i(U, \bar{U})_L$ , for any  $P \in 1 + TL[T]$ , where  $I$  is the ideal in a divided power series ring over  $W(k)$  considered in *op.cit.*. We note that since  $P$  is monic the formal Laurent series  $1/P(T) = \sum_{n \geq 0} a_n T^n$  has positive radius of convergence, so there is some  $A$  such that  $\text{ord}_p(a_n) \geq -nA$  for  $n \gg 0$ . This implies the convergence of the series  $\sum a_n \Phi^n$  on  $I \otimes_{W(k)} H_{\text{HK}}^i(U, \bar{U})_L$ , which gives an inverse of  $P(\Phi)$ .

Continuing to follow [10], we have

**Proposition 2.2.8** *If  $(T, \bar{T})$  is the base-change of  $(U, \bar{U})$  to a finite Galois extension  $K'/K$  with Galois group  $G$ , then the natural map  $f : (T, \bar{T}) \rightarrow (U, \bar{U})$  induces a quasi-isomorphism*

$$f^* : \mathbb{R}\Gamma_{\text{syn}, L, P}(U, \bar{U}, r) \xrightarrow{\sim} \mathbb{R}\Gamma(G, \mathbb{R}\Gamma_{\text{syn}, L, P}(T, \bar{T}, r)).$$

*Proof* This follows by exactly the same proof as in *op.cit.*: the proof proceeds by constructing compatible maps between the various variants of crystalline, de Rham, and Hyodo–Kato cohomology, and these all remain compatible after extending scalars to  $L$ .

**Remark 2.2.9** We are principally interested in the case  $L = K$ , but it seems to be easier to prove base-change compatibility in  $K$  for a fixed  $L$ . It seems eminently natural that if  $K'/K$  is totally ramified and both fields are finite over  $\mathbb{Q}_p$  then we should get a quasi-isomorphism

$$\mathbb{R}\Gamma_{\text{syn}, P}(U, \bar{U}, r)_K \xrightarrow{\sim} \mathbb{R}\Gamma(G, \mathbb{R}\Gamma_{\text{syn}, P}(T, \bar{T}, r)_{K'}),$$

but this does not seem to be so easy to prove, and it is not needed for the calculations below.

## 2.3 h-sheafification

We now sheafify in the  $h$ -topology. We write  $\mathcal{S}_{L, P}(r)$  for the sheafification of  $(U, \bar{U}) \mapsto \mathbb{R}\Gamma_{\text{syn}, L, P}(U, \bar{U}, r)$ , and we define

$$\mathbb{R}\Gamma_{\text{syn}, L, P}(X_h, r) = \mathbb{R}\Gamma(X_h, \mathcal{S}_{L, P}(r)).$$

**Proposition 2.3.1** *For any arithmetic pair  $(U, \bar{U})$  that is fine, log-smooth over  $V^\times$  and of Cartier type, the canonical maps*

$$\mathbb{R}\Gamma_{\text{syn}, L, P}(U, \bar{U}, r) \rightarrow \mathbb{R}\Gamma_{\text{syn}, L, P}(U_h, r)$$

*are quasi-isomorphisms.*

*Proof* This is exactly the generalization to our setting of Proposition 3.16 of *op.cit.*. The long and highly technical proof fortunately carries over verbatim to general  $P$ .

**Proposition 2.3.2** *There is a “ $P$ -syntomic descent spectral sequence”*

$$E_2^{ij} = H_{\text{st}, L, P}^i \left( D^j(X_h)(r) \right) \Rightarrow H_{\text{syn}, L, P}^{i+j}(X_h, r),$$

where as above  $D^j(X_h) \cong \mathbf{D}_{\text{pst}} \left( H^j(X_{\bar{K}, \text{ét}}, \mathbb{Q}_p) \right)$  is the  $(\varphi, N, G_K)$ -module defined by the comparison between Hyodo–Kato and de Rham cohomology. This spectral sequence is compatible with extension of  $L$  (where defined) and change of  $P$ . Moreover, it is compatible with cup-products, where the cup-product on the terms  $E_2^{ij}$  is given by the construction of Sect. 1.3.

*Proof* For the existence of the spectral sequence, see Proposition 3.17 of *op.cit.*. The compatibility with cup-products is immediate from the definition of the cup-product on  $P$ -syntomic cohomology (see Sect. 2.4 below for explicit formulae).

Note that the spectral sequence implies that the cohomology groups  $H_{\text{syn},L,P}^i(X_h, r)$  are zero if  $i > 2 \dim X + 2$ , and if  $K/\mathbf{Q}_p$  is a finite extension, then the cohomology groups are finite-dimensional  $L$ -vector spaces (with dimension bounded independently of  $P$ ). Moreover, the spectral sequence obviously degenerates at  $E_3$ , and this gives a 3-step filtration on the groups  $H_{\text{syn},L,P}^i(X_h, r)$ :

**Definition 2.3.3** We write  $\text{Fil}^m H_{\text{syn},L,P}^i(X_h, r)$  for the 3-step decreasing filtration on  $H_{\text{syn},L,P}^i(X_h, r)$  induced by the  $P$ -syntomic spectral sequence. Concretely, we have

$$\begin{aligned} \text{Fil}^0 / \text{Fil}^1 &= \ker \left( H_{\text{HK}}^i(X_h)_L^{P(\Phi_r)=0, N=0} \cap \text{Fil}^r H_{\text{dR}}^i(X_h) \rightarrow \frac{H_{\text{HK}}^{i-1}(X_h)_L}{\text{im } P(\Phi_{r-1}) + \text{im}(N)} \right), \\ \text{Fil}^1 / \text{Fil}^2 &= \frac{\left\{ (x, y, z) \in H_{\text{HK}}^{i-1}(X_h)_L \oplus H_{\text{HK}}^{i-1}(X_h)_L \oplus H_{\text{dR}}^{i-1}(X_h) / \text{Fil}^r : Nx = P(\Phi_{r-1})y \right\}}{\left\{ (P(\Phi_r)x, Nx, \iota_{\text{dR}}(x)) : x \in H_{\text{HK}}^{i-1}(X_h)_L \right\}}, \\ \text{Fil}^2 / \text{Fil}^3 &= \text{coker} \left( H_{\text{HK}}^{i-1}(X_h)_L^{P(\Phi_r)=0, N=0} \cap \text{Fil}^r H_{\text{dR}}^{i-1}(X_h) \rightarrow \frac{H_{\text{HK}}^{i-2}(X_h)_L}{\text{im } P(\Phi_{r-1}) + \text{im}(N)} \right). \end{aligned}$$

*Remark 2.3.4* It seems natural to conjecture that the “knight’s move” maps

$$H_{\text{st},L,P}^0(D^j(X_h)(r)) \rightarrow H_{\text{st},L,P}^2(D^{j-1}(X_h)(r))$$

should be zero for smooth proper  $X$ , extending the conjecture for syntomic cohomology formulated in Remark 4.10 of [10]. We do not know if this conjecture holds in general, but the applications below will all concern cases where either the source or the target of this map is zero.

## 2.4 Explicit cup product formulae

We now use Proposition 2.2.6 to give an explicit formula for the cup product

$$H_{\text{syn},L,P}^i(X_h, r) \times H_{\text{syn},L,Q}^j(X_h, s) \longrightarrow H_{\text{syn},L,P \star Q}^{i+j}(X_h, r+s),$$

which generalizes the description of the cup-product on the complexes  $C_{\text{st},L,P}^\bullet(D)$  given in the previous section.

A class  $\eta \in H_{\text{syn},P}^i(X_h, r)$  is represented by the following data:

$$\begin{aligned} u &\in \mathbb{R}\Gamma_{\text{HK}}^{B,i}(X_h)_L, & v &\in \text{Fil}^r \mathbb{R}\Gamma_{\text{dR}}^i(X_h), \\ w, x &\in \mathbb{R}\Gamma_{\text{HK}}^{B,i-1}(X_h)_L, & y &\in \mathbb{R}\Gamma_{\text{dR}}^{i-1}(X_h)_L, \\ z &\in \mathbb{R}\Gamma_{\text{HK}}^{B,i-2}(X_h)_L \end{aligned}$$

which satisfy the relations

$$\begin{aligned} du &= 0, & dv &= 0, \\ dw &= P(\Phi)u, & dx &= Nu, & dy &= \iota_{\text{dR}}^B(u) - v, \\ dz &= Nw - P(q\Phi)x. \end{aligned}$$

Let  $\eta' = [u', v'; w', x', y'; z']$  be a corresponding representation of  $\eta' \in H_{\text{syn}, L, Q}^j(X_h, s)$ . We want to find explicit formulae for the class  $\eta'' = \eta \cup \eta'$ .

As before, fix polynomials  $a(t, s)$  and  $b(t, s)$  such that

$$(P \star Q)(ts) = a(t, s)P(t) + b(t, s)Q(s),$$

and  $\lambda \in K$ .

**Proposition 2.4.1** *The cup-product is given by  $\eta'' = [u'', v''; w'', x'', y''; z'']$ , where*

$$\begin{aligned} u'' &= u \cup u', \\ v'' &= v \cup v', \\ w'' &= a(\Phi, \Phi)(w \cup u') + (-1)^i b(\Phi, \Phi)(u \cup w'), \\ x'' &= (x \cup u') + (-1)^i (u \cup x'), \\ y'' &= y \cup \iota_{\text{dR}}^B \left( (1 - \lambda)u' + \lambda v' \right) + (-1)^i \iota_{\text{dR}}^B \left( \lambda u + (1 - \lambda)v \right) \cup y', \\ w'' &= a(q\Phi, \Phi)(z \cup u') - (-1)^i a(\Phi, q\Phi)(w \cup x') + (-1)^i b(q\Phi, \Phi)(x \cup w') \\ &\quad - b(\Phi, q\Phi)(u \cup z'). \end{aligned}$$

Here, we write e.g.  $a(\Phi, \Phi)(w \cup u')$  for the image of  $a(\Phi_1, \Phi_2)(w \otimes u') \in \mathbb{R}\Gamma_{\text{HK}}^{B, i}(X_h)_L \otimes \mathbb{R}\Gamma_{\text{HK}}^{B, j-1}(X_h)_L$  under the cup product map into  $\mathbb{R}\Gamma_{\text{HK}}^{B, i+j-1}(X_h)_L$ , where  $\Phi_1$  and  $\Phi_2$  are the  $q$ -power Frobenius maps of the two factors.

*Proof* This is simply a translation of the formulae of Sect. 1.3 into the setting of complexes.

## 2.5 Pushforward maps and compact supports

We now extend to our setup the constructions of Déglise's Appendix B to [10].

**Theorem 2.5.1** *Let  $f : X \rightarrow Y$  be a smooth proper morphism of smooth  $K$ -varieties, of relative dimension  $d$ . Then there are pushforward maps*

$$f_* : H_{\text{syn}, L, P}^i(X_h, r) \rightarrow H_{\text{syn}, L, P}^{i+2d}(X_h, r + d),$$

*which are compatible with change of  $L$  and change of  $P$  in the obvious sense.*

*Proof* Each of the sheaves  $\mathcal{S}_{L, P}(r)$  is  $h$ -local, by definition. They are also  $\mathbf{A}^1$ -local: this is virtually immediate from the corresponding result for the underlying Hyodo–Kato and de Rham cohomology theories, as in Prop 5.4 of *op.cit.*. The same methods also yield the projective bundle theorem.

Hence we can consider the direct sum of  $h$ -sheaves given by

$$\mathcal{S}_{L, P} := \bigoplus_r \mathcal{S}_{L, P}(r).$$

Cup-product gives us maps  $\mathcal{S}_L(1) \otimes \mathcal{S}_{L, P}(r) \rightarrow \mathcal{S}_{L, P}(r+1)$ , where  $\mathcal{S}_L(1)$  is defined using the polynomial  $1 - T$ ; this gives the direct sum  $\mathcal{S}_{L, P}$  the structure of a Tate  $\Omega$ -spectrum. The same argument as in *op.cit.* now gives pushforward maps (and the projection formula for these holds by construction). It is clear that this argument is compatible with change of  $L$  and of  $P$ .

The same argument also gives compactly-supported cohomology complexes:

**Theorem 2.5.2** *There exist compactly-supported cohomology complexes  $\mathbb{R}\Gamma_{\text{syn},L,P,c}(X_h, r)$ , contravariantly functorial with respect to proper morphisms and covariantly functorial (with a degree shift) with respect to smooth morphisms; and there is a functorial morphism*

$$\mathbb{R}\Gamma_{\text{syn},L,P,c}(X_h, r) \rightarrow \mathbb{R}\Gamma_{\text{syn},L,P}(X_h, r)$$

*which is an isomorphism for proper  $X$ .*

*The compactly-supported cohomology has a descent spectral sequence*

$$E_{ij}^2 = H_{\text{st},L,P}^i \left( D_c^j(X_h)(r) \right) \Rightarrow H_{\text{syn},L,P,c}^{i+j}(X_h, r)$$

*where  $D_c^j(X_h) = \mathbf{D}_{\text{pst}}(H_{\text{ét},c}^i(X_{\overline{K},\text{ét}}, \mathbf{Q}_p))$ .*

Finally, we have a projection formula:

**Theorem 2.5.3** *There are cup-products*

$$\mathbb{R}\Gamma_{\text{syn},L,P,c}(X_h, r) \times \mathbb{R}\Gamma_{\text{syn},L,Q}(X_h, s) \rightarrow \mathbb{R}\Gamma_{\text{syn},L,P \star Q,c}(X_h, r + s),$$

*and for  $f$  a smooth proper morphism, we have the projection formula*

$$f_*(\alpha) \cup \beta = f_*(\alpha \cup f^*\beta).$$

*Proof* This follows from the construction of the pushforward map, cf. [9].

### 3 Application to computation of regulators

We'll now apply the constructions above to give formulae describing the syntomic regulator map  $H_{\text{mot}}^i(X, j) \rightarrow H_{\text{syn}}^i(X_h, j)$ , where  $X$  is a product of copies of an affine curve  $Y$ . More specifically, we will give formulae in the following cases:

- $X = Y$  is a single curve and we are given a class in  $H_{\text{mot}}^2(X, 2)$  that is the cup product of two units on  $Y$ ;
- $X = Y^2$  is the product of two curves, and we are given a class in  $H_{\text{mot}}^3(X, 2)$  that is the pushforward of a unit along the diagonal embedding  $Y \hookrightarrow Y^2$ ;
- $X = Y^3$  is the product of three curves and we consider the class in  $H_{\text{mot}}^4(X, 2)$  given by the cycle class of the diagonal  $Y \hookrightarrow Y^3$ .

In the case where  $Y$  has a smooth model over  $\mathbf{Z}_p$  these regulators have been described using finite-polynomial cohomology (in [4, 6, 7] respectively); and these have been applied in proving explicit reciprocity laws for Euler systems (in [2, 3, 8] respectively). We use the generalization of finite-polynomial cohomology to arbitrary varieties described in the preceding section to extend this description to the case of arbitrary smooth curves over  $K$ . However, our results are less complete, in that we only obtain a full description of the image of the regulator in a “convenient” quotient of the cohomology of  $X$  in the sense of Sect. 1.4. (In other words, we allow  $X$  to have bad reduction, but we project to a quotient of its cohomology which looks like the cohomology of a variety with good reduction.)

### 3.1 Syntomic regulators

Now let us consider the following general setting:  $X$  is a smooth connected  $d$ -dimensional affine  $K$ -variety, and  $z \in H_{\text{mot}}^{d+1}(X, j)$  for some  $j$ . We have an étale realization map

$$r_{\text{ét}} : H_{\text{mot}}^{d+1}(X, j) \longrightarrow H_{\text{ét}}^{d+1}(X_{K, \text{ét}}, \mathbf{Q}_p(j)).$$

The Hochschild–Serre descent spectral sequence gives a map

$$H_{\text{ét}}^{d+1}(X_{K, \text{ét}}, \mathbf{Q}_p(j)) \rightarrow H^0(K, H^{d+1}(X_{\bar{K}, \text{ét}}, \mathbf{Q}_p(j)));$$

but the latter group is zero (because an affine variety of dimension  $d$  has étale cohomological dimension  $d$ ) and thus we obtain a map

$$H^{d+1}(X_{K, \text{ét}}, \mathbf{Q}_p(j)) \rightarrow H^1(K, H^d(X_{\bar{K}, \text{ét}}, \mathbf{Q}_p(j))).$$

Theorem B of [10] shows that we have  $r_{\text{ét}} = \rho_{\text{syn}} \circ r_{\text{syn}}$ , where

$$r_{\text{syn}} : H_{\text{mot}}^{d+1}(X, j) \longrightarrow H_{\text{syn}}^{d+1}(X_{K, h}, j)$$

is the syntomic realization map, and

$$\rho_{\text{syn}} : H_{\text{syn}}^{d+1}(X_{K, h}, j) \rightarrow H_{\text{ét}}^{d+1}(X_{K, \text{ét}}, \mathbf{Q}_p(j))$$

is the syntomic period morphism of Theorem 2.1.1 above. Theorem 2.1.2 shows that we have a diagram

$$\begin{array}{ccc} H_{\text{syn}}^{d+1}(X_{K, h}, j) & \xrightarrow{\rho_{\text{syn}}} & H_{\text{ét}}^{d+1}(X_{K, \text{ét}}, \mathbf{Q}_p(j)) \\ \downarrow & & \downarrow \\ H_{\text{st}}^1(D^d(X_h)(j)) & \xrightarrow{-\exp_{\text{st}}} & H^1(K, H^d(X_{\bar{K}, \text{ét}}, \mathbf{Q}_p(j))) \end{array}$$

where  $D^d(X_h) = \mathbf{D}_{\text{pst}}(H^d(X_{\bar{K}, \text{ét}}, \mathbf{Q}_p))$ , the vertical maps are induced by the syntomic and Hochschild–Serre descent spectral sequences, and in the bottom horizontal map,  $\exp_{\text{st}}$  denotes the generalized Bloch–Kato exponential for the de Rham  $G_K$ -representation  $H^d(X_{\bar{K}, \text{ét}}, \mathbf{Q}_p(j))$ .

*Remark 3.1.1* The appearance of  $-\exp_{\text{st}}$ , rather than  $\exp_{\text{st}}$ , in the bottom horizontal map is due to the sign  $v = (-1)^p v'$  (for  $p = 1$ ) in Step 2 of [10, Theorem 2.17].

Now let  $D$  be a quotient of  $D^d(X_h)(j)$  (in the category of weakly-admissible  $(\varphi, N, G_K)$ -modules) which is crystalline and convenient, in the sense of Sect. 1.4 above. Then we have isomorphisms

$$H_{\text{st}}^1(D) \cong H_{\text{st}, K, 1-T}^1(D) \cong D_K / \text{Fil}^0 = (\text{Fil}^0 D^*(1)_K)^*,$$

where  $D^*(1)$  is the Tate dual of  $D$ , which is a submodule of  $D_c^d(X_h)(d+1-j)$ . As we saw above, for any class  $\eta \in \text{Fil}^0 D^*(1)_K$ , we may choose a polynomial  $P$  such that  $\eta \in H_{\text{st}, K, P}^0(D^*(1))$  and  $P(1) \neq 0$ ,  $P(q^{-1}) \neq 0$ ; and the natural perfect pairing  $(D_K / \text{Fil}^0) \times (\text{Fil}^0 D^*(1)_K) \rightarrow K$  coincides with the cup-product

$$H_{\text{st}, K}^1(D) \times H_{\text{st}, K, P}^0(D^*(1)) \rightarrow H_{\text{st}, K, P}^1(\mathbf{Q}_p(1)) \cong K.$$

We now relate this to cup-products in  $P$ -syntomic cohomology.

**Definition 3.1.2** If  $P(1) \neq 0$  and  $P(q^{-1}) \neq 0$ , then we denote by

$$\mathrm{tr}_{X, \mathrm{syn}, K, P} : \frac{H_{\mathrm{syn}, X, P, c}^{2d+1}(X_h, d+1)}{\mathrm{Fil}^2} \xrightarrow{\cong} H_{\mathrm{st}, K, P}^1(\mathbf{Q}_p(1)) \cong K$$

the isomorphism given by the descent spectral sequence and Proposition 1.4.2 above.

**Theorem 3.1.3** Let  $D$  be a convenient crystalline quotient of  $D^d(X_h)(j)$ , and let  $\eta \in \mathrm{Fil}^0 D^*(1)_K \subseteq \mathrm{Fil}^{d+1-j} H_{\mathrm{dR}, c}^d(X_h)$ . If

$$\mathrm{pr}_D : H_{\mathrm{syn}}^{d+1}(X_h, j) \longrightarrow H_{\mathrm{st}}^1(D^d(X_h)(j)) \longrightarrow H_{\mathrm{st}}^1(D) \cong D_K / \mathrm{Fil}^0$$

is the natural projection, then for any  $z \in H_{\mathrm{mot}}^{d+1}(X, j)$ , any polynomial  $P$  such that  $P(\Phi)(\eta) = 0$  and  $P(1) \neq 0$ ,  $P(q^{-1}) \neq 0$ , and any class

$$\tilde{\eta} \in H_{\mathrm{syn}, K, P, c}^d(X_h, d+1-j)$$

lifting  $\eta$ , we have

$$\eta(\mathrm{pr}_D(r_{\mathrm{syn}}(z))) = \mathrm{tr}_{X, \mathrm{syn}, K, P}(r_{\mathrm{syn}, K}(z) \cup \tilde{\eta})$$

where  $r_{\mathrm{syn}, K}(z)$  is the image of  $r_{\mathrm{syn}}(z)$  under the natural map  $H_{\mathrm{st}}^1(D^d(X_h)(j)) \rightarrow H_{\mathrm{st}, K, 1-T}^1(D^d(X_h)(j))$ .

*Proof* This is immediate from Proposition 1.4.3 and the compatibility of the  $P$ -syntomic descent spectral sequence with cup-products.

Using the above formula together with the projection formula for cup-products (Theorem 2.5.3), we obtain the following consequences (in which  $j = 2$  and  $1 \leq d \leq 3$ ). For  $\eta \in H_{\mathrm{dR}, c}^d(X/K)$  satisfying the hypotheses of Theorem 4.2, we write  $\lambda_\eta$  for the map  $H_{\mathrm{syn}}^{d+1}(X_h, j) \rightarrow K$  given by the composition of  $\mathrm{pr}_D$  and pairing with  $\eta$ .

**Proposition 3.1.4** If  $d = 3$  and  $z = \sum_i n_i [Z_i] \in H_{\mathrm{mot}}^4(X, 2)$  is the class of a codimension 2 cycle with smooth components, and  $\eta$  is as in Theorem 3.1.3, then we have

$$\lambda_\eta(r_{\mathrm{syn}}(z)) = \sum_i n_i \mathrm{tr}_{Z_i, \mathrm{syn}, K, P}(\iota_i^*(\tilde{\eta})),$$

where  $\iota_i$  is the inclusion of  $Z_i$  into  $X$ .

*Proof* Cf. [4, Theorem 1.2]. By the compatibility of motivic and syntomic pushforward maps, we have

$$r_{\mathrm{syn}, K}(z) = \sum_i n_i (\iota_i)_* (\mathbf{1}_{Z_i}),$$

where  $\mathbf{1}_{Z_i} \in H_{\mathrm{syn}, K}^0(Z_i, 0)$  is the identity class. By the projection formula, we have

$$r_{\mathrm{syn}, K}(z) \cup \tilde{\eta} = \sum_i n_i (\iota_i)_* (\mathbf{1}_{Z_i} \cup \iota_i^* \tilde{\eta}).$$

Since the maps  $\mathrm{tr}_{X, \mathrm{syn}, K, P}$  are compatible with pushforward (as is clear by comparison with their de Rham analogues), the result is now immediate from Theorem 2.5.3.

**Proposition 3.1.5** If  $d = 2$  and  $z = \sum_i (Z_i, u_i) \in H_{\mathrm{mot}}^3(X, 2)$ , where  $Z_i$  are codimension 1 cycles and  $u_i \in \mathcal{O}(Z_i)^\times$ , and  $\eta$  is as in Theorem 3.1.3, then we have

$$\lambda_{\eta}(r_{\text{syn}}(z)) = \sum_i n_i \operatorname{tr}_{Z_i, \text{syn}, K, P}(r_{\text{syn}, K}(u_i) \cup \iota_i^*(\tilde{\eta})),$$

where  $\iota_i$  is the inclusion of  $Z_i$  into  $X$ . Here,  $r_{\text{syn}}$  is the composition

$$\mathcal{O}(Z_i)^{\times} = H_{\text{mot}}^1(Z_i, 1) \longrightarrow H_{\text{syn}}^1(Z_i, 1).$$

*Proof* Again, by the compatibility of motivic and syntomic pushforwards, we have

$$r_{\text{syn}, K}(z) = \sum_i (\iota_i)_*(r_{\text{syn}, K}(u_i)),$$

and the result is immediate from Theorem 2.5.3.

**Proposition 3.1.6** *If  $d = 1$  and  $z = \sum_i \{u_i, v_i\} \in H_{\text{mot}}^2(X, 2)$ , where  $\{u_i, v_i\}$  is the Steinberg symbol of two units  $u_i, v_i \in \mathcal{O}(X)^{\times}$ , and  $\eta$  is as in Theorem 3.1.3, then we have*

$$\lambda_{\eta}(r_{\text{syn}}(z)) = \sum_i \operatorname{tr}_{X, \text{syn}, K, P}(r_{\text{syn}, K}(u_i) \cup r_{\text{syn}, K}(v_i) \cup \tilde{\eta}).$$

*Proof* Observe that the image of  $\{u_i, v_i\}$  in  $H_{\text{syn}}^2(X, 2)$  is equal to  $r_{\text{syn}, K}(u_i) \cup r_{\text{syn}, K}(v_i)$ .

### 3.2 The cohomological triple symbol

We now define a “triple symbol” attached to three de Rham classes on a curve  $Y$ , which is closely related to the global triple index of [7]. Let  $Y$  be a connected smooth affine curve over  $K$ .

Let us choose a class  $\eta \in H_{\text{dR}, c}^1(Y/K)$ ; and let  $\omega_1, \omega_2 \in \operatorname{Fil}^1 H_{\text{dR}}^1(Y/K)$ .

**Assumption 1** The following conditions are satisfied:

- There is a crystalline  $(\varphi, N, G_K)$ -submodule  $D_0 \subseteq D_c^1(X_h)$  such that  $\eta \in (D_0)_K$ , and similarly  $D_1, D_2 \subseteq D^1(X_h)$  such that  $\omega_i \in (D_i)_K$  for  $i = 1, 2$ .
- There are polynomials  $P_0, P_1, P_2 \in 1 + TK[T]$  such that  $P_0(\Phi)$  annihilates  $\eta$  and  $P_i(q^{-1}\Phi)$  annihilates  $\omega_i$  for  $i = 1, 2$ , so that

$$\eta \in H_{\text{st}, K, P_0, c}^0(D_c^1(Y_{K, h})), \quad \omega_1 \in H_{\text{st}, K, P_1}^0(D^1(Y_{K, h})(1)), \quad \omega_2 \in H_{\text{st}, K, P_2}^0(D^1(Y_{K, h})(1)).$$

- The class  $\omega_1$  is in the kernel of the “knight’s move” map

$$H_{\text{st}, K, P_1}^0(D^1(Y_{K, h})(1)) \rightarrow H_{\text{st}, K, P_1}^2(D^0(Y_{K, h})(1)),$$

and similarly for  $\omega_2$ .

- The polynomial  $P_0 \star P_1 \star P_2$  does not vanish at 1 or  $q^{-1}$ .
- We have  $\eta \cup \omega_1 = \eta \cup \omega_2 = 0$  as elements of  $H_{\text{dR}, c}^2(Y/K) \cong K$ .

*Remark 3.2.1* We have

$$H_{\text{st}, K, P}^2(D^0(Y_{K, h})(1)) = H_{\text{st}, K, P}^2(\mathbf{Q}_p(1)) = \frac{K}{P(1)K},$$

so if  $P_1(1) \neq 0$ , then  $\omega_1$  is automatically in the kernel of the “knight’s move” map. In particular, this applies if  $\omega_1$  is pure of weight 1. On the other hand, if  $\omega_1 = \operatorname{dlog} u$  for a unit  $u$ , then we have no choice but to take  $P_1(1) = 0$ , but the existence of the syntomic regulator  $r_{\text{syn}}$  and its compatibility with the de Rham regulator forces  $\omega_1$  to be in the kernel of this map since we know it is in the image of  $H_{\text{syn}}^1(Y_h, 1)$ .

If all three de Rham classes are pure of weight 1, then  $P_0 \star P_1 \star P_2$  has all its roots of weight  $-1$ , so in particular it is non-vanishing at 1 and  $q^{-1}$ .



**Definition 3.2.2** Under the above assumptions, we define the *triple symbol*  $[\eta; \omega_1, \omega_2] \in K$  by the formula

$$[\eta; \omega_1, \omega_2] = \mathrm{tr}_{Y, \mathrm{syn}, P_0 \star P_1 \star P_2} (\tilde{\eta} \cup \tilde{\omega}_1 \cup \tilde{\omega}_2)$$

where  $\tilde{\eta} \in H_{\mathrm{syn}, K, P_0, c}^1(Y_{K, h}, 0)$  and  $\tilde{\omega}_i \in H_{\mathrm{syn}, K, P_i}^1(Y_{K, h}, 1)$  are liftings of  $\eta$  and the  $\omega_i$ .

**Proposition 3.2.3** *The above quantity is independent of the choice of liftings and of the polynomials  $P_i$ .*

*Proof* Firstly, we note that the natural map  $H_{\mathrm{syn}, K, P_0, c}^1(Y_{K, h}, 0) \rightarrow H_{\mathrm{dR}, c}^1(Y/K)$  is an isomorphism, since the degree 0 compactly-supported cohomology is zero. Thus the class  $\tilde{\eta}$  is uniquely defined.

If  $\tilde{\omega}_1$  is a class in  $H_{\mathrm{syn}, K, P_1}^1$  lifting  $\omega_1$ , and  $[u, v; w, x, y; z]$  is a representative of  $\tilde{\omega}_1$  in  $\mathbb{R}\Gamma_{\mathrm{syn}, K, P_1}^1$  (with  $z = 0$ , necessarily, for degree reasons), then any other choice of lifting can be represented by  $[u, v; w + \lambda, x + \mu, y; z]$  for some constants  $\lambda, \mu \in K$  (with  $\mu = 0$  unless  $P_1(1) = 0$ ). From the definition of the cup-product, one sees that varying  $\mu$  has no effect on  $\tilde{\eta} \cup \tilde{\omega}_1 \cup \tilde{\omega}_2$ ; while varying  $\lambda$  changes the cup product by a multiple of  $\eta \cup \omega_2$ , which is zero by assumption. Similarly, the assumption that  $\eta \cup \omega_1 = 0$  implies that the cup-product is independent of the choice of lifting of  $\omega_2$ .

So  $[\eta; \omega_1, \omega_2]$  is well-defined for a fixed choice of polynomials  $P_i$ . However, both the cup-product and the map  $\mathrm{tr}_{Y, \mathrm{syn}, P_0 \star P_1 \star P_2}$  are compatible with change of the polynomials  $P_i$ , so the symbol  $[\eta; \omega_1, \omega_2]$  is also independent of these choices.

**Remark 3.2.4** We may also carry out the same construction if  $Y$  is projective, rather than affine, if we add the assumption that  $P_0(1) \neq 0$  and  $P_0(q) \neq 0$ ; this assumption implies that there is still a unique lifting of  $\eta$  to  $H_{\mathrm{syn}, K, P_0, c}^1(Y_{K, h}, 0) = H_{\mathrm{syn}, K, P_0}^1(Y_{K, h}, 0)$ . In particular, this holds if  $P_0$  is pure of weight 1.

### 3.3 Products of curves

We now use the theory of the previous section to reformulate special cases of Propositions 3.1.4–3.1.6 in terms of the triple symbol. We continue to assume that  $Y$  is a connected smooth affine curve over  $K$ .

**Proposition 3.3.1** *Let  $\eta \in H_{\mathrm{dR}, c}^1(Y/K)$ , and let  $\omega_1, \omega_2 \in \mathrm{Fil}^1 H_{\mathrm{dR}}^1(Y/K)$  be classes lying in the parabolic subspace (the image of  $H_{\mathrm{dR}, c}^1$  in  $H_{\mathrm{dR}}^1$ ). Suppose that the triple symbol  $[\eta; \omega_1, \omega_2]$  is defined.*

*Let  $X = Y \times Y \times Y$  and let  $z \in H_{\mathrm{mot}}^4(X, 2)$  be the class of the diagonal embedding  $Y \hookrightarrow Y \times Y \times Y$ . If  $\mu$  denotes the class  $\eta \otimes \omega_{1, c} \otimes \omega_{2, c} \in \mathrm{Fil}^2 H_{\mathrm{dR}, c}^3(X/K)$ , where  $\omega_{i, c}$  are any liftings of  $\omega_i$  to compactly-supported cohomology, then*

$$\lambda_\mu(r_{\mathrm{syn}}(z)) = [\eta; \omega_1, \omega_2].$$

**Proposition 3.3.2** *Let  $\eta \in H_{\mathrm{dR}, c}^1(Y/K)$  and let  $\omega \in \mathrm{Fil}^1 H_{\mathrm{dR}}^1(Y/K)$  be a class lying in the parabolic subspace. Let  $u \in \mathcal{O}(Y)^\times$ , and suppose that the triple symbol  $[\eta; \omega_1, \mathrm{dlog} u]$  is defined.*

*Let  $X = Y \times Y$  and let  $z \in H_{\mathrm{mot}}^3(X, 2)$  be the pushforward of  $u \in H_{\mathrm{mot}}^1(Y, 1) \cong \mathcal{O}(Y)^\times$  along the diagonal embedding  $Y \hookrightarrow Y \times Y$ . If  $\mu$  denotes the class  $\eta \otimes \omega_c \in \mathrm{Fil}^1 H_{\mathrm{dR}, c}^2(X/K)$ , where  $\omega_c$  is any lifting of  $\omega$  to compactly-supported cohomology, then*

$$\lambda_\mu(r_{\mathrm{syn}}(z)) = [\eta; \omega, \mathrm{dlog} u].$$

**Proposition 3.3.3** *Let  $\eta \in H_{\mathrm{dR},c}^1(Y/K)$ . Let  $u, v \in \mathcal{O}(Y)^\times$ , and suppose that the triple symbol  $[\eta; \mathrm{dlog} u, \mathrm{dlog} v]$  is defined.*

*Let  $z \in H_{\mathrm{mot}}^2(X, 2)$  be the cup-product of the classes  $u, v \in H_{\mathrm{mot}}^1(Y, 1) \cong \mathcal{O}(Y)^\times$ . Then*

$$\lambda_\eta(r_{\mathrm{syn}}(z)) = [\eta; \mathrm{dlog} u, \mathrm{dlog} v].$$

### 3.4 An alternative description of the triple symbol

We conclude by giving an alternative, equivalent description of the symbol  $[\eta; \omega_1, \omega_2]$ , which we hope may be useful in relating our results to  $p$ -adic modular forms (as in the calculations of [8] in the good-reduction case).

The assertion that  $(P_0 \star P_1 \star P_2)(q^{-1}) \neq 0$  implies that  $(P_1 \star P_2)(q^{-1} \Phi^{-1})$  acts bijectively on the kernel of  $P_0(\Phi)$ , so that  $\frac{1}{(P_1 \star P_2)(q^{-1} \Phi^{-1})} \eta$  is well-defined.

**Proposition 3.4.1** *Suppose that  $\tilde{\omega}_1 \cup \tilde{\omega}_2 \bmod \mathrm{Fil}^2$  is represented by the class*

$$w, x, y] \in H_{\mathrm{st},K,P_1 \star P_2}^1(D^1(Y_h)(2)).$$

*Then we have*

$$[\eta; \omega_1, \omega_2] = \eta \cup y - \left( \frac{1}{(P_1 \star P_2)(q^{-1} \Phi^{-1})} \eta \right) \cup w.$$

*Proof* From the definition of the cup product, we see that it respects the 3-step filtration on  $P$ -syntomic cohomology, and the cup-products induced on the graded pieces coincide with the usual de Rham cup products. Since  $\tilde{\omega}_1 \cup \tilde{\omega}_2$  lies in  $\mathrm{Fil}^1$ , and the trace isomorphism factors through  $\mathrm{Fil}^1 / \mathrm{Fil}^2$ , we obtain the above compatibility using (2).  $\square$

This takes a particularly simple form if  $\eta$  is a  $\Phi$ -eigenvector, say  $\Phi(\eta) = \alpha\eta$ ; then we have

$$[\eta; \omega_1, \omega_2] = \eta \cup \left( y - \frac{1}{(P_1 \star P_2)(q^{-1} \alpha^{-1})} w \right).$$

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